calculation. The following expression, up to third-order terms exact, for the rotation

$$\left(\frac{\psi}{r}\right) = \frac{\dot{w}+v}{r^2(1-Pr/EF)} - \frac{1}{6}\frac{(\dot{w}+v)^3}{r^3(1-Pr/EF)^3} \tag{3}$$

is easily obtained from the differential geometry of the buckled ring and can now be used to write the potential functional Eq. (2) in terms of the radial (w) and the tangential (v) displacement component. Adding to this the energy stored in the assumed Winkler foundation, the total potential functional of the elastically embedded ring is obtained. Expanding the displacement functions as

$$w = \sum_{i}^{\infty} a_i \cos i\phi \tag{4a}$$

and

$$v = \sum_{i}^{\infty} b_i \sin i\phi \tag{4b}$$

setting $Pr/EF \cong 0$ and omitting the details of the elementary discrete perturbation procedure,[4] the initial post buckling associated with the smallest critical load (in the case of a relatively large foundation constant c, where large waves number $i \geqq 4$ can occur)

$$P_{\min}{}^c = 2(cEI)^{1/2}/r \tag{5}$$

is given by

$$\frac{P}{P^c} = 1 + \frac{1}{4}\left(\frac{cr^4}{EI}\right)^{1/2} \xi^2 \tag{6}$$

where the associated wave number is $i = r(c/EI)^{1/4}$, $\xi = a_s/r$ is the perturbation parameter and a_s is the maximal buckling amplitude. Unlike the corresponding strut problem, the ring in an elastic foundation possesses a stable symmetric point of bifurcation and is, therefore, imperfection insensitive.

References

[1] Lekkerkerker, J. G., "On the Stability of an Elastically Supported Beam Subjected to Its Smallest Buckling Load," *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen*, Ser. B, Vol. 65, p. 190, 1962.

[2] El Naschie, M. S., *Zeitschrift fuer Angewandte Mathematik und Mechanik*, Vol. 54, 1974, p. 677.

[3] Koiter, W. T., "Elastic Stability and Post Buckling Behaviour," edited by R. E. Langer, *Proceedings of the Symposium on Nonlinear Problems*, Vol. 257, 1963.

[4] Thompson, J. M. T., "A General Theory for the Equilibrium and Stability of Discrete Conservative Systems," *Zeitschrift fuer Angewandt Mathematik und Physik*, Vol. 20, June 1969, p. 797.

Two-Dimensional Inverse Conduction Problem—Further Observations

MURRAY IMBER*
Polytechnic Institute of New York, Brooklyn, N.Y.

IN a recent series of papers, Refs. 1 and 2, the author develops an analytical method for temperature extrapolation in heat conduction systems. For the one-dimensional configuration, a solution is generated whereby the differential equation is satisfied in a manner which does not restrict its extrapolation property. From the two prescribed interior conditions

$$T(x_1, t) = T_1(t) \quad \text{and} \quad T(x_2, t) = T_2(t) \tag{1}$$

Received July 15, 1974.
Index category: Heat Conduction.

* Professor, Mechanical Engineering Department. Member AIAA.

the temperature may be predicted outside the spatial range, $x_2 - x_1$. Since it is presumed that Eq. (1) represents thermocouple responses, the actual traces are approximated by two temporal power series of degree, n. Dependent upon the direction of the predication process, one of the interior conditions is reformulated in the Laplace transform plane so that the positive arguments in the solution are suppressed. Therefore, the relationship between the thermocouples is written as

$$\bar{T}_1(s) = \bar{T}_2(s) \sum_{m=1}^{M} A_m \exp\left[-m(x_2 - x_1)(s/\alpha)^{1/2}\right] \tag{2}$$

where s is the complex number and α the thermal diffusivity. The summation in the transformed temperature relationship, Eq. (2), constitutes an enabling function whose coefficients A_m are evaluated from the inverse transform. The explicit expression for the temperature predictor is shown in Ref. 1, and it should be noted that this solution fulfils the requirements that the temperatures must be known throughout a closed interior region. In the one-dimensional case the two planes $x = x_1$ and $x = x_2$ constitute the closed region. Obviously, the solution will return the values for the thermocouple responses dictated by Eq. (1). It is also the solution to the direct conduction problem for a slab of thickness, $x_2 - x_1$, with boundary conditions represented by Eq. (1).

Before proceeding to a discussion of the two-dimensional case, it has been demonstrated that the one-dimensional prediction process is very accurate. Its success may be attributed to the limited number of inter-related approximations that must be made. Experience gained through use of the method indicates that a ninth degree polynomial approximation for Eq. (1) was sufficient. In Eq. (2) any value of M may be selected; however as this value approached the degree of the power series the results were substantially improved. For high accuracy, it is recommended that the value $M = n$ be used. There is little difficulty in the numerical evaluation of the coefficient A_m, since these are obtained from a, $n \times n$, matrix which is insensitive to roundoff errors. As a precaution, sentinel thermocouples may be positioned outside the range, $x_2 - x_1$, and the extrapolation values may be compared with these. The closer the correspondence, the more successful the prediction method.

Considerably more complicated is the extrapolation program for a two-dimensional system. In this instance, the temperature responses are only known at discrete locations around the perimeter of an internally enclosed region. Representing the thermocouple traces as

$$T(x_i, y_j, t) = \sum_{n=1}^{n} b_n{}^{ij} t^n \tag{3}$$

there can be a large number of data points which the analytical solution must agree with. As shown in Ref. 2, a solution is obtained by point-matching. The essence of the method is to satisfy four perimeter positions simultaneously for each increment in time. To illustrate what can occur, suppose the thermocouples are purposefully positioned on an interior rectangle, the thermocouple rectangle. Initially, a solution is obtained with the temperature traces at the corners as the only input data. For this case, the corner solution, the expression not only satisfies the differential equation by construction, but it also returns the values for the temperature traces at the four corners. It does not however follow that the desired solution has been obtained. Additional thermocouple information positioned around the perimeter of the thermocouple rectangle should be compared with the theoretical results. If the resultant deviations are all small, then it can be anticipated that the prediction process will be successful. Positioning of sentinel thermocouples, provides an additional check.

The corner solution involves the least number of approximations, and the related matrix for the determination of the coefficients in the enabling functions is modest in size; consequently the matrix solution is insensitive. It is interesting to note that the corner solution has no parallel as a solution to a direct conduction system. Specifying the boundary conditions

Table 1 Deviation of internal predicted temperatures, $N = 2$

Time hr	(0.15, 0.3) $n = 5$	(0.15, 0.7) $n = 5$	(0.3, 0.15) $n = 5$	(0.7, 0.15) $n = 5$
0.10	2.12×10^{-2}	3.59×10^{-2}	8.45×10^{-2}	-4.40×10^{-2}
0.15	-0.12×10^{-2}	-0.29×10^{-2}	-1.03×10^{-2}	-0.43×10^{-2}
0.20	0.19×10^{-2}	0.22×10^{-2}	0.33×10^{-2}	-0.30×10^{-2}
0.25	0.09×10^{-2}	0.01×10^{-2}	-0.60×10^{-2}	0.16×10^{-2}
0.30	0.32×10^{-2}	0.43×10^{-2}	1.08×10^{-2}	-0.78×10^{-2}

at the corners of rectangular solid is an insufficient requirement, since additional information must be provided as to the behavior between the corners.

Should the corner solution prove to be unacceptable, then it is necessary to introduce additional perimeter thermocouple responses as a set of four, one for each face. As shown in Ref. 2, the original corner solution is amended to incorporate the new thermocouple inputs. The numerical results are then compared with the nonutilized perimeter positions, and the method can be repeated to embrace as many thermocouple inputs as desired. There is, however, a practical limitation. The related matrix equation for the enabling coefficients grows as $4(n)$ per set of thermocouple inputs. If N represents the number of matched points excluding the corners, the size of the involved matrix would be $4(n)(N)$. In other words, a two point match with a fifth degree approximation polynomial entails the solution of 40 equations with 40 unknowns. Furthermore, the sensitivity grows with the size of the matrix. A partial reduction of this effect may be achieved by re-arranging the coefficient matrix so that the diagonal elements contain the largest values. It is, however, recommended that the matrix size be kept as small as possible.

Tables 1–3 indicate the extrapolation results for a rectangular solid with sides (1 ft, π ft). Three of the rectangle's faces are maintained at zero temperature, and the remaining face, $x = 0$, varies linearly with time, i.e., $T(0, y, t) = t$. The thermocouple rectangle is a square of length 0.4 ft with the nearest corner located at (0.3 ft, 0.3 ft).

Table 1 represents a comparison of the predicted and sentinel traces at the four positions denoted by the braces. For a two point match coupled with a fifth degree approximation, the backward extrapolation is not successful for the earliest time increment, $t = 0.10$ hr. In the main, there is some increase in accuracy over the one point match shown in Ref. 2. On the basis of these results, backward prediction to the relevant faces should be satisfactory, except for, $t = 0.10$ hr. Table 2 is a compilation of the surface predicted values, and as anticipated good agreement occurs for all values except, $t = 0.10$ hr. A comparison of the deviations contained in Tables 1 and 2 reveals an unexpected result. Apparently, the magnitude of the deviation grows as the extrapolation distance increases. Table 3 presents the computational results for forward extrapolation. The preceding observations also apply to this situation. At the earliest time, forward extrapolation is unconvincing. The remaining tabulated results are quite good.

Table 2 Surface prediction temperatures, $N = 2$

Time hr	(0, 0.3) $n = 5$	(0, 0.7) $n = 5$	(0.3, 0) $n = 5$	(0.7, 0) $n = 5$
0.10	0.0646	0.0394	−0.1100	0.0584
0.15	0.1550	0.1539	0.0226	−0.0087
0.20	0.1988	0.1905	−0.0111	0.0098
0.25	0.2526	0.2459	0.0175	−0.0045
0.30	0.2948	0.2795	−0.0534	0.0350

Table 3 Deviation of prediction temperatures, $N = 2$

Time hr	(0.85, 0.3) $n = 5$	(0.85, 0.7) $n = 5$	(1.0, 0.3) $n = 5$	(1.0, 0.7) $n = 5$
0.10	-1.37×10^{-2}	-0.47×10^{-2}	-2.13×10^{-2}	-0.78×10^{-2}
0.15	0.20×10^{-2}	0.08×10^{-2}	0.54×10^{-2}	0.15×10^{-2}
0.20	0.00×10^{-2}	0.02×10^{-2}	0.06×10^{-2}	0.00×10^{-2}
0.25	0.12×10^{-2}	0.06×10^{-2}	-0.41×10^{-2}	0.11×10^{-2}
0.30	-0.05×10^{-2}	0.00×10^{-2}	0.16×10^{-2}	-0.09×10^{-2}

As previously mentioned, the coefficient matrix for the enabling function coefficients is large, 40 by 40, with elements that are very small. Accordingly, precise determination of the appropriate values for the first time interval is very difficult. Since the analytical solution should return the perimeter data inputs, the computations reveal this is generally so except at, $t = 0.10$ hr. For the initial time, the deviation may be as high as 200%; however the magnitude of the quantities involved are extremely small. It therefore follows that the prediction process will do well for longer times, since the error contribution from the inaccuracy associated with the initial time value is negligible.

In conclusion, the two-dimensional extrapolation method is considerably more sensitive than the one-dimensional case. It is advisable to position sentinel thermocouples whenever possible. By placing the thermocouple rectangle in close proximity to the region of extrapolation, error growth may be contained. The matrix size should be kept as small as possible. This can be achieved by using a smaller order approximation polynomial coupled with a minimum number of matched points. In situations where this is not possible, some sacrifice in accuracy must be accepted.

References

[1] Imber, M. and Khan, J., "Prediction of Transient Temperature Distributions with Embedded Thermocouples," *AIAA Journal*, Vol. 10, No. 6, June 1972, pp. 784–789.

[2] Imber, M., "A Temperature Extrapolation Mechanism for Two Dimensional Heat Flow," *AIAA Journal*, Vol. 12, No. 8, Aug. 1974, pp. 1089–1093.

Analytical Solution for Shock Wave Precursors

H. F. NELSON*
University of Missouri-Rolla, Rolla, Mo.

I. Introduction

In the last few years the literature has contained several reports of experimental and theoretical investigations of the electron and excited state populations in shock wave precursors. Some of the theoretical investigations involve the complete nonequilibrium shock wave structure behind the shock[1–4] while others assume a blackbody emitter behind the shock.[5–7] Measurements of precursor populations have been reported by Weymann,[8] Holmes and Weymann,[9] Teshima et al.,[10] and Lederman and Wilson.[11] This Note presents a simple analytical solution for the precursor electron and excited state number density due to radiative processes and compares the results with

Received July 18, 1974.
Index categories: Radiatively Coupled Flows and Heat Transfer; Shock Waves and Detonations; Radiation and Radiative Heat Transfer.

* Thermal Radiative Transfer Group, Mechanical and Aerospace Engineering; presently on leave, Engineer, Liquid and Propulsion Section, Jet Propulsion Laboratory. Member AIAA.